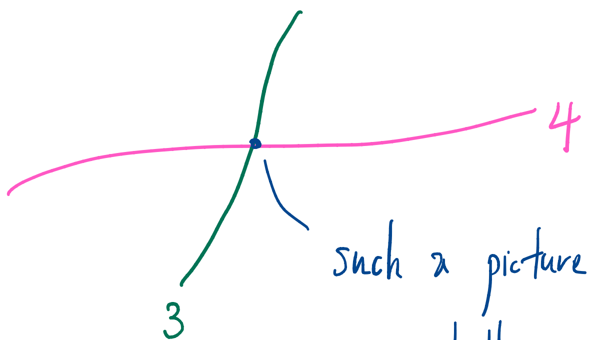
Math 53: Multivariable Calculus **Sections 212, 213**

Worksheet for 2020-09-21

Problem 1. Here are some conceptual questions on the gradient and directional derivatives.

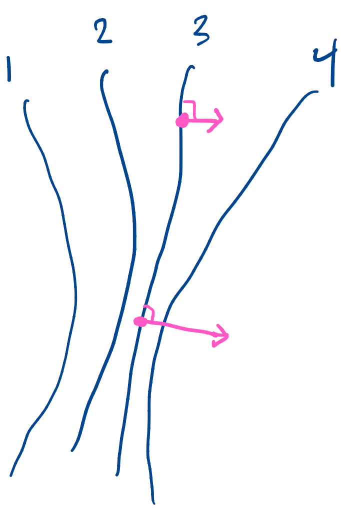
(a) Is it possible for different level sets of a function to intersect?
(b) How are the direction and magnitude of the gradient vector related to level sets?
(c) If $\mathbf{r}(t)$ is a curve contained in the surface $f(x, y, z) = 0$, how are the vectors $\mathbf{r}'(3)$ and $\nabla f(\mathbf{r}(3))$ related? (Are they parallel? Orthogonal? Something else?)
(d) Fix a function $f(x, y)$, a number c, and a point (a, b) where $\nabla f(a, b) \neq \mathbf{0}$. How many unit vectors $\mathbf{u}$ are such that $D_{\mathbf{u}}(a, b) = c$? Hint: the answer depends on $|c|$.

(a) No.



Such a picture would imply that the function is both equal to 3 and 4 at this point, which is nonsense.

(b).



The gradient is perpendicular to level sets and points towards higher level sets. The gradient's magnitude is larger when level sets are closer together, since that means the function is increasing more rapidly there.

(c) Just as in (b), they are orthogonal. You can also see this using the chain rule:

since $f(\vec{r}(t)) = 0$ always, $\frac{d}{dt}\left[f(\vec{r}(t))\right] = 0$

$\|$ See 2(a)

$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ so these two vectors are orthogonal.

(d) $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$

$= |\nabla f(a,b)|\, |\vec{u}| \cos\theta$ where θ is the angle between $\nabla f(a,b)$ and $\vec{u}$

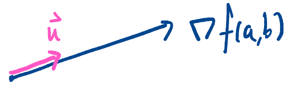
$= |\nabla f(a,b)| \cos\theta$ (it is between 0 and π)

So $\cos\theta = c / |\nabla f(a,b)|$.

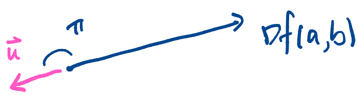
If $|c| > |\nabla f(a,b)|$, then the equation above has no solution for θ, so there are <u>no</u> possible $\vec{u}$.

If $|c| \le |\nabla f(a,b)|$, then the equation above has one solution for θ:

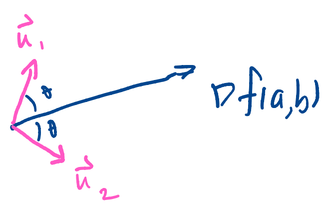
If $c = |\nabla f(a,b)|$ then $\theta = 0$ and there is <u>one</u> possible $\vec{u}$:



If $c = -|\nabla f(a,b)|$ then $\theta = \pi$ and there is <u>one</u> possible $\vec{u}$:



If $|c| < |\nabla f(a,b)|$ then $0 < \theta < \pi$ and there are <u>two</u> possible $\vec{u}$:



Problem 2.

(a) Let $f(x, y)$ be a function on $\mathbb{R}^2$ and $\mathbf{r}(t)$ be an arc-length parametrized path in $\mathbb{R}^2$ (in other words, $|\mathbf{r}'(t)| = 1$ for all t).

Use the chain rule to show that

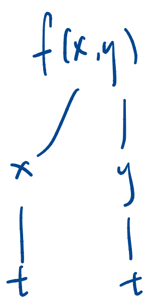
$$D_{\mathbf{r}'(t)} f(\mathbf{r}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t)).$$

(b) Use the path $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ to compute $f_y(1, 0)$ for $f(x, y) = \cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$.

Note: $\vec{r} = \langle x, y \rangle$

(a) $$\frac{d}{dt} f(\vec{r}(t)) = f_x(\vec{r}(t))\frac{dx}{dt} + f_y(\vec{r}(t))\frac{dy}{dt}$$

$$= \underbrace{\langle f_x(\vec{r}(t)), f_y(\vec{r}(t)) \rangle}_{= \nabla f(\vec{r}(t))} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}_{= \vec{r}\,'(t)},$$ which we assumed to be a unit vector.



$$= D_{\vec{r}\,'(t)} f(\vec{r}(t)).$$

(b) $|\vec{r}\,'(t)| = |\langle -\sin t, \cos t \rangle| = 1$ so the situation in (a) is applicable.

$\vec{r}\,'(0) = \langle 0, 1 \rangle$

$$f_y(1,0) = D_{\langle 0,1 \rangle} f(1,0) = D_{\vec{r}\,'(0)} f(\vec{r}(0))$$

$$= \frac{d}{dt} f(r(t)) \Big|_{t=0} = \frac{d}{dt} \cos^{-1}\left(\frac{\cos^2 t - \sin^2 t}{1}\right) \Big|_{t=0}$$

$$= \frac{d}{dt} \cos^{-1}(\cos(2t)) \Big|_{t=0} = \frac{d}{dt}(2t) \Big|_{t=0} = \boxed{2}$$